Tutorial 5.

Preliminary:

Increasing Annuities: $(Ia)_{\overline{n}|i} = X$.

$$(Ia)_{\overline{n}|i} = \frac{\ddot{a}_{\overline{n}|i} - nv^n}{i}$$

$$(Ia)_{\overline{\infty}|i} = \frac{1}{i^2} + \frac{1}{i}$$

Present-Value.

$(Ia)_{\overline{n}|i} = v + 2v^2 + \cdots + nv^n$, then $(1+i)X = 1 + 2v + \cdots + nv^{n-1}$.

$iX = 1 + v + \cdots + v^{n-1} - nv^n$. $\quad X = \frac{1 + \cdots + v^{n-1} - nv^n}{i} = \frac{\ddot{a}_{\overline{n}|i} - nv^n}{i} = (Ia)_{\overline{n}|i}$.

Perpetuity:

When $n \to \infty$, $(Ia)_{\overline{\infty}|i} = \lim_{n\to\infty} \frac{\ddot{a}_{\overline{n}|i} - nv^n}{i} = \frac{1}{id}$, because $\lim_{n\to\infty} \ddot{a}_{\overline{n}|i} = \frac{1}{d}$, $\lim_{n\to\infty} nv^n = 0$.

$= \frac{1}{i^2} + \frac{1}{i}$

A-V: $(Is)_{\overline{n}|i} = \frac{\ddot{s}_{\overline{n}|i} - n}{i}$

Decreasing Annuities:

$(Da)_{\overline{n}|i} = nv + (n-1)v^2 + \cdots + v^n = \frac{n - a_{\overline{n}|i}}{i}$.

$(Ds)_{\overline{n}|i} = \frac{n(1+i)^n - s_{\overline{n}|}}{i} = (Da)_{\overline{n}|i} \cdot (1+i)^n$.

Exercise:

2.3.3.

Jeff: $\quad X = 30\, a_{\overline{\infty}|k\%} = \frac{30}{k\%}$ $\quad$ "perpetuity-immediate".

Jason: ~~"increasing"~~

53 $\quad$ 53(1+k%) $\quad$ 53(1+k%)^9

0 $\quad$ 1 $\quad \cdots$ $\quad$ 10

$X = 53v + 53\cdot(1+k\%)v^2 + \cdots + 53(1+k\%)^9 v^{10}$

$= 53v\,(1 + (1+k\%)v + \cdots + (1+k\%)^9 v^9)$

since $v = \frac{1}{1+k\%}$, then $(1+k\%)v = 1$.

hence, $X = 53v \cdot 10 = \frac{530}{1+k\%}$. $\quad \frac{530}{1+k\%} = \frac{30}{k\%} \Rightarrow k\% = 6\%$.

2.3.5.

$j = 6\%$ $\quad$ monthly rate $(1+i)^{12} = 1.06\%$. $\Rightarrow i = \sqrt[12]{1.06\%} - 1 = 0.49\%$.

20 years

$P_0 \quad P_1 \quad P_2 \quad P_3 \quad P_{19}$

0 $\quad$ 1 $\quad$ 2 $\quad$ 3 $\cdots$ 19 $\quad$ 20

Each Year

$R(1+i\%)^{n-1}$

0 $\quad$ 1 $\quad$ 2 $\quad \cdots$ $\quad$ 11 $\quad$ 12

$$P_n = R \cdot 1.032^n \cdot a_{\overline{12}|i}$$

$100{,}000 = P_0 + P_1 v_j + P_2 v_j^2 + \cdots + P_{19} v_j^{19} = R\cdot 1.032^0 \cdot a_{\overline{12}|i} + \cdots R\cdot 1.032^{19}\cdot a_{\overline{12}|i} \cdot v_j^{19}$

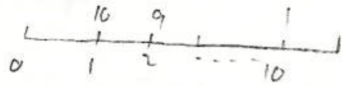
$= R\, a_{\overline{12}|i} \cdot (1 + 1.032\cdot v_j + \cdots + 1.032^{19} v_j^{19}) = R \cdot \frac{1 - v_i^{12}}{i} \cdot \frac{1 - \left(\frac{1.032}{1.06}\right)^{20}}{1 - \frac{1.032}{1.06}}$

$\Rightarrow R = 547.9$.

2.3.18.

Present Value for Annuity 1.

$$(Da)_{\overline{10}|i} = \frac{10 - a_{\overline{10}|i}}{i}$$



"$(Da)_{\overline{10}|i}$"

Present Value for Annuity 2.

The difference.

"$11 a_{\overline{\infty}|i}$"

$-$

"$(Da)_{\overline{10}|i}$"

$$\Rightarrow \quad 11 a_{\overline{\infty}|i} - (Da)_{\overline{10}|i} = 2(Da)_{\overline{10}|i}$$

$$11 a_{\overline{\infty}|i} = 3(Da)_{\overline{10}|i}$$

$$\frac{11}{i} = 3 \cdot \frac{10 - a_{\overline{10}|i}}{i}$$

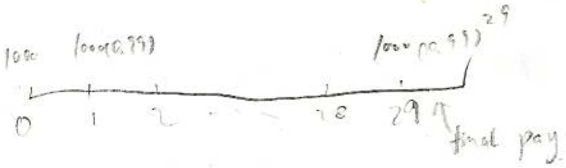
$$\Rightarrow a_{\overline{10}|i} = \frac{19}{3} \Rightarrow i = 0.093.$$

$$(Da)_{\overline{10}|i} = \frac{10 - \frac{19}{3}}{0.093} = 39.4.$$

2-3-3, 2-3-5, 2-3-18 ~~2-3-19~~. Problem Set 2-3-2, 2-3-4, 2-3-7, 2-3-8, 2-3-11, 2-3-12.

2-3-2.

(i)



$$A\cdot V = 1000(1.01)^{29} + 1000(0.99)(1.01)^{28} + \cdots + 1000(0.99)^{29}$$

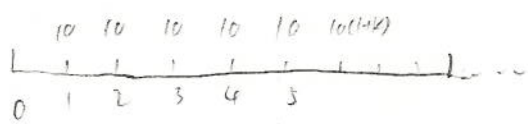
$$= 1000(1.01)^{29}\left(1 + \frac{0.99}{1.01} + \left(\frac{0.99}{1.01}\right)^2 + \cdots \left(\frac{0.99}{1.01}\right)^{29}\right)$$

$$= 1000(1.01)^{29} \cdot \frac{1-\left(\frac{0.99}{1.01}\right)^{30}}{1-\frac{0.99}{1.01}} = 30,407$$

(ii) $AV = 1000(1.05)^{29} \cdot \dfrac{1-\left(\frac{0.99}{1.05}\right)^{30}}{1-\frac{0.99}{1.05}} = 59,704.$

(iii) $AV = 1000(1.1)^{29} \dfrac{1-\left(\frac{0.99}{1.1}\right)^{30}}{1-\frac{0.99}{1.1}} = 151,906.$

2-3-4.



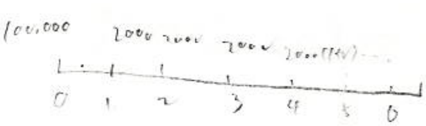
$i = 9.2\%$ $\quad v = \dfrac{1}{1+9.2\%}$, $\quad d = \dfrac{9.2\%}{1+9.2\%} = 1-v.$

$$PV = 10v + 10v^2 + \cdots + 10v^5 + 10(1+k)v^6 + 10(1+k)v^7 + \cdots$$

$$= 10a_{\overline{5}|9.2\%} + 10(1+k)v^6(1+v+v^2+\cdots) = 10a_{\overline{5}|9.2\%} + 10(1+k)v^6 \cdot \frac{1}{1-v} = 10a_{\overline{5}|9.2\%} + 10(1+k)v^6 \cdot \frac{1}{d} = 167.5.$$

$\Rightarrow k = 0.04.$

2-3-7.



$i = 4.5\%.$

$$PV = 100,000 = 2000\left(v + v^2 + v^3 + (1+r)v^4 + (1+r)v^5 + (1+r)v^6 + (1+r)^2v^7 + \cdots\right)$$

$$= 2000(v+v^2+v^3)\left(1 + v^3(1+r) + v^6(1+r)^2 + \cdots + v^{3n}(1+r)^n + \cdots\right)$$

$$= 2000\, a_{\overline{3}|4.5\%} \cdot \frac{1-(1+r)^n v^{3n}}{1-(1+r)v^3} = \frac{2000\, a_{\overline{3}|4.5\%}}{1-(1+r)v^3} \qquad v = \frac{1}{1+4.5\%}.$$

$\Rightarrow r = 0.0784.$

2.3-8

(a)

$18{,}000\,(1+(1+4\%)+\cdots+(1+4\%)^{36}) = 18{,}000\; s_{\overline{37}|4\%} = 1{,}470{,}640.$

average salary $= \dfrac{1{,}470{,}640}{37} = 39{,}747.$

Pension $= 0.70 \times 39{,}747 = 27{,}823.$

(b) $0.25 \times 37 \times 39{,}747 = 36{,}766$

(c) $18{,}000\,[(1+4\%)^{36}+\cdots+(1+4\%)^{27}] \times \dfrac{1}{10} = 62{,}312.$

Pension $= (0.25)\times 37 \times 62{,}312 = 57{,}639.$

(d) annual ~~payment~~ deposit is $18{,}000\,(1+4\%)^{n}\cdot 3\%$

$AV = 18{,}000\cdot 6\%\cdot(1+6\%)^{36} + 18{,}000\cdot(1+4\%)\cdot 6\%\cdot(1+6\%)^{35}+\cdots+18{,}000\,(1+4\%)^{36}\cdot 3\%$

$= 6\% \times 18{,}000 \times (1.06)^{36}\cdot\left(\dfrac{1-\left(\frac{1.04}{1.06}\right)^{37}}{1-\frac{1.04}{1.06}}\right) = 247{,}845$

$247{,}845 = X\cdot \ddot{a}_{\overline{20}|6\%} \Rightarrow X = 19{,}774.$

2.3.11.

Sandy: $PV = 100v + (100+10)v^2 + \cdots + (100+10n)v^n+\cdots$

$= 90(v+v^2+\cdots) + 10v + 20v^2+\cdots$

$= 90\,a_{\overline{\infty}|i} + 10\cdot(Ia)_{\overline{\infty}|i} = 90\cdot\frac{1}{i} + 10\left(\frac{1}{i}+\frac{1}{i^2}\right)$

Danny: $PV = 180\,\ddot{a}_{\overline{\infty}|i} = \dfrac{180}{d} = \dfrac{180(1+i)}{i}$

$\dfrac{180(1+i)}{i} = \dfrac{90}{i} + 10\left(\frac{1}{i}+\frac{1}{i^2}\right) \Rightarrow i = 10.2\%.$

2.3.12.

$X = 2(Ia)_{\overline{60}|j}$, j is monthly rate, 3-month rate is $\frac{9\%}{4} = 2.25\%$.

$(1+j)^3 = 1+2.25\% \Rightarrow j = 0.007444$

$X = 2\cdot\dfrac{\ddot{a}_{\overline{60}|j} - 60v_j^{60}}{j} = 2729.$